**Introduction**

The subject of this course is mathematical models. Our general aim is the justification of mathematical models. We have an interest to obtaining of mathematical models, its analysis and practical solving. We will consider it by means of classical and generalized methods. The standard classical method is based on the classical mathematical analysis. The generalized methods use distributions theory and topological technics. We will consider stationary heat transfer as the unique example because its simplicity and the natural physical sense.

**1. Classical models of the mathematical physics**

## 1.1. Mathematical analysis of a physical phenomenon

**Three steps of research:**

* Derivation of the model (physicist)
* Analysis of the model (theorist mathematician)
* Solving of the model (application mathematician)



Fig. 1.1. Scheme of research.

We consider some physical phenomenon. We would like to obtain its mathematical model.

**Standard steps of obtaining mathematical models:**

1. Definition of an *elementary volume* for the analyzing system.
2. Determination of *balance relations* in this volume by physic low.
3. *Passing to the limit* at these balance relations with obtaining a mathematical model.

The state equation is the result. This is an equation with respect to the state function. State equations with corresponding initial and boundary conditions are classical mathematical models.



Fig. 1.2. Derivation of mathematical physics models.

## 1.2. Obtaining of the mathematical model

We consider the stationary heat transfer phenomenon as an example. Let us consider one dimensional body with length *L*. Suppose that there exists a source of the heat and the temperature of the body equals to zero on its boundary.

We would like to know the distribution of the temperature with respect to the length of the body.

Determine the change of the quantity of the heat *q* on an interval [*x*,*x+h*]

**** (1.1)

The known function *f* characterizes the source of the heat here. The function *f* determines the heat on the unit interval under thus source. So the integral of the right side of the last equality is the heat of the given interval of the body under the source. It is equal to the difference between quantity of the heat in begin and end of the interval by equality (1.1).

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Fig. 1.3. Change on the heat quantity.

By Fourier low the heat flux in a point *x* is proportional to difference between the temperature in begin of the interval and in its end. Then it is inverse proportional to the length of the interval. Besides, the heat moves from the domain with large temperature to the domain with small temperature. So we obtain the formula

****  (1.2)

where *u* is the temperature, and *k* is the heat conductivity of the body.

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Fig. 1.4. Fourier low.

Suppose the function *u* is twice continuously differentiable in the given set. After passing to the limit in the equalities (1.1), (1.2) as *h* → 0 we get *non homogeneous stationary heat equation*

 (1.3)

We add also the boundary conditions

*u*(0) = 0*, u*(*L*) = 0, (1.4)

namely the temperature of the body on the boundary of the interval is equal to zero. The boundary problem (1.3), (1.4) is *the classical mathematical model of our phenomenon*.

## 1.3. Classical solution of the system

We consider twice continuously differentiable function *u*, which satisfies the equalities (1.3), (1.4).

**Definition 1.1**. *The twice continuously differentiable function* *u on the interval* (0,*L*), *which satisfies the equalities* (1.3), (1.4), *is called* ***the classical solution*** *of this boundary problem*.

Hence obtaining of the mathematical model is associated to the notion of the classical solution of the mathematical physics problem. Then we have an interest to the qualitative and quantitative analysis of this mathematical model. The qualitative analysis is at first the proof of solvability of the system. It is necessary to prove that the boundary problem (1.3), (1.4) has a classical solution under some property of the known functions *f* and *k*. This result can be obtained with using of the differential equations theory.

The final result is the practical solving of the system. It can be realized with using approximated methods.

## 1.4. Approximate solution of the system

The analytical formula of the dependence of the solution of the problem (the temperature *u* for our case) from the independent variable (space coordinate *x* for our situation) for all value of the parameters of the system (length of the body *L*, source of the body *f*, its heat conductivity *k*) can be obtained only for easiest problems. We can find only approximate solution of the problem for the real situation. There exist a lot of approximate methods of solving for the problem (1.3), (1.4). We chose *finite difference method*. It uses the division of the given domain to parts and approximation of the derivative by the corresponding difference.

Consider a function **** Let it be differentiable. Then we have the equality

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where  as . If the value *h* is smooth enough, we have the approximate equality

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So we can find

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This formula of approximate differentiation is called “forward difference”. Analogically from the equality

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we deduce the approximate equality

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Then

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This formula of approximate differentiation is called “back difference”.

We divide the interval (0,*L*) to the *M* equal parts. Determine the step *h = L*/*M* and the points **** Determine standard difference operators

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by equalities

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where ****

The classical solution of the problem (1.3), (1.4) is twice continuous differentiability function. In this case the differential equation (1.3) can be approximate by the difference equation

**** (1.5)

at the points *xi*, where



We add also the boundary conditions

*u*0 = 0, *uM* = 0. (1.6)

The system of linear algebraic equations (1.5), (1.6) can be solved be means of standard methods (for example, marching method). So we find all values *ui*, namely the grid function. But our solution is the function with continuous argument. Therefore we use the linear interpolation (see Fig. 1.5)

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It satisfies the equality



Besides it is equal to zero on the boundary of our domain.

May be we prove the convergence *uh* → *u* in the class of the twice continuously differentiable functions as *h* → 0. Then the limit *u* is the classical solution of our boundary problem. It is the substantiation of the numerical method and the basis of the practical solution of the problem.

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Fig. 1.5. Linear interpolation of the grid function.

Note that the properties of classical solution are used for all three steps of the research (see Fig. 1.6).



Fig. 1.6. Assumption for the state function under the classical method.

## 1.5. Validity of the classic method

We obtained our mathematical model by means of passing to the limit. However there is the question, if this method is valid? Our analysis is correct if the state function *u* is twice continuous differentiable. Then we have next question, why it does satisfy this property?

We could use physical reasons. The temperature of the body must be smooth function. But our body can be non-homogeneous. Our heat source can have different properties. We don’t know, if the temperature will be twice continuous differentiable for these cases. Besides our object is mathematical model, but not physical body. We don’t know if properties of the phenomenon and its model are same. So we need to use only mathematical reasons.

We could use results of the differential equations theory. We could prove that the solution of our boundary problem is twice continuous differentiable under some assumptions of the known functions *k* and *f*. However this result can be obtained after the determination of the mathematical model. But we cannot any possibility to analyze the equation before its definition.

Our results are strange enough. The determination of the model uses the properties of its state function. But it is necessary to have the model for the analysis its state function (see Fig. 1.7).



Fig. 1.7. Relations between obtaining of the mathematical model and properties of state functions.

We will try to correct these results. We try to replace the classical method of the analysis on the generalized method.